Journal of Computational and Applied Mathematics 235 (2011) 1790–1797

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A numerical scheme for periodic travelling-wave simulations in some nonlinear dispersive wave models

J. Álvarez [a], A. Durán [b,*]

[a] Departamento de Matemática Aplicada, E.T.S. de Ingenieros Industriales, Universidad de Valladolid, Paseo del Cauce S/N, E-47011 Valladolid, Spain

[b] Departamento de Matemática Aplicada, E.T.S.I. de Telecomunicación, Universidad de Valladolid, Campus Miguel Delibes, Camino del cementerio S/N E-47011 Valladolid, Spain

ARTICLE INFO

Article history:
Received 1 July 2010
Received in revised form 14 July 2010

MSC:
65M20
65M99
35Q53
76B25

Keywords:
Periodic travelling waves
Petviashvili's method
Conserved quantities
Conservative numerical methods
Projection methods

ABSTRACT

A numerical method for simulating periodic travelling-wave solutions of some nonlinear dispersive wave equations is proposed. The construction of the scheme is based on an efficient computation of the elements that characterize these solutions: the initial profile and the velocity of the wave.

1. Introduction

In this paper we introduce a numerical method for simulating travelling-wave solutions of the periodic problem for nonlinear dispersive wave equations of the general form

$$u_t + f(u)_x - Mu_x = 0, \quad x \in \mathcal{R}, \ t > 0 \tag{1}$$

where $u = u(x, t)$ is a $2L$-periodic, real-valued function of the two real independent variables x, t; f is a smooth, real-valued function of u, representing a nonlinear term, and M is a linear, nonnegative, formally self-adjoint operator, characterized as a Fourier multiplier operator by its symbol

$$\widehat{Mv}(\xi) = \alpha(\xi)\widehat{v}(\xi), \tag{2}$$

where $\widehat{\ }$ denotes the Fourier transform. Equations of the form (1) appear in many models concerning the propagation of small-amplitude, nonlinear, dispersive long waves; see e.g. [1–3] and references therein as a modest representation of the literature on the initial value problem for (1). Important cases include the generalized KdV equation ($f(s) = s^p/p$, $p \geq 2$, $M = -\partial_{xx}$), the generalized Benjamin–Ono equation ($f(s) = s^p/p$, $p \geq 2$, $M = -\mathcal{H}\partial_x$, where $\mathcal{H}$ stands for the Hilbert transform) and the Benjamin equation ($f(s) = s^2$, $M = -\gamma_1\mathcal{H}\partial_x - \gamma_2\partial_{xx}$, for some parameters γ_1, γ_2). See [3] for more examples.

* Corresponding author.
E-mail addresses: joralv@eis.uva.es (J. Álvarez), angel@mac.uva.es (A. Durán).

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doi:10.1016/j.cam.2010.07.015

Periodic travelling-wave solutions of (1) are periodic functions of the form $\phi(x-ct)$, for $c>0$, representing the velocity of the wave, and they play a relevant role in the models (see e.g. [4]). In general, explicit expressions for these solutions cannot be obtained by analytical techniques and a numerical treatment for their computation is necessary. Traditional requirements for the numerical integrators, classical properties like high order of convergence, might not be sufficient to get a good simulation, especially for long times [5–7]. Our main idea is to consider also the approximation of these waves in a geometric sense. The numerical method that we describe here is then focused on the elements that characterize these solutions. First we need to implement an efficient computation of the profile ϕ, combining a suitable spatial discretization and an iterative procedure. On the other hand, a correct simulation of the velocity determines the selection of the time integrator.

The paper is structured as follows. In Section 2 we make some hypotheses on (1) and recall some analytical properties of the equations under study that are relevant for our work. The numerical method is treated in Section 3. It includes a description of the spatial discretization, the iterative technique for approximating the initial profiles and the time integration. These ideas are numerically illustrated in Section 4.

2. Preliminaries

Several hypotheses on the nonlinear term f and the symbol α are assumed.

(H1) f is a polynomial of the form $f(z)=a_pz^p+\cdots+a_2z^2$ with $a_p>0$, $a_j\geq 0$, $j=2,\ldots,p-1$, for some $p\geq 2$.
(H2) $\alpha:\mathbb{R}\to\mathbb{R}$ is continuous, even, nonnegative with $\alpha(0)=0$.
(H3) α is monotone increasing on $[0,\infty)$ and there exists $m>(p-1)/p$ such that $\liminf_{\xi\to\infty}\alpha(\xi)/|\xi|^m>0$.

Note that some conditions on $\alpha(\xi)$, like (H3), are related to the nonlinear term f [8]. Also, in some cases the operator M has to be adapted to the periodic case, considering, for instance, Eq. (2) for the Fourier coefficients.

Some relevant properties of the periodic problem for (1) relate to our study. The first one is the existence of an at least three invariants of the problem

$$I(u)=\int_{-L}^{L}u(x,t)\mathrm{d}x, \tag{3}$$

$$V(u)=\int_{-L}^{L}u^2(x,t)\mathrm{d}x, \tag{4}$$

$$H(u)=\int_{-L}^{L}\frac{1}{2}(u(x,t)Mu(x,t))-F(u(x,t))\mathrm{d}x, \tag{5}$$

where $F'=f$, $F(0)=0$. They are functionals preserved by sufficiently smooth $2L$-periodic solutions of (1). The quantity (5) enables the Hamiltonian structure of the periodic problem

$$u_t=\mathcal{J}\delta H(u),\quad u\in X,$$

where δ denotes the variational derivative, $\mathcal{J}=\partial_x$ and X is the Sobolev space $X=H^s_{\mathrm{per}}$ of $2L$-periodic functions for some $s\geq 1/2$ and the usual norm

$$\|u\|_s=\left(\sum_{k=-\infty}^{\infty}(1+|k|^2)^s|\hat{u}_k|^2\right)^{1/2},$$

where $\hat{u}_k$ denotes the k-th Fourier coefficient of u. Hypotheses (H1)–(H3) are assumed to guarantee the existence of solutions of (1) [9,10] in X and they include the important cases cited above.

A second relevant property that we mention here is the existence of periodic travelling-wave solutions [10]. They are periodic solutions of a permanent form $u(x,t)=\phi(x-ct)$, travelling with a constant velocity $c>0$. The profile $\phi=\phi_c$ is $2L$-periodic and, in order for u to be a solution of (1), must satisfy the equation

$$\mathcal{J}(\delta H(\phi_c)+c\delta V(\phi_c))=0,$$

which can be written as

$$\delta H(\phi_c)+c\delta V(\phi_c)=A\delta I(\phi_c), \tag{6}$$

or, in terms of the elements of (1),

$$M\phi_c-f(\phi_c)+c\phi_c=A, \tag{7}$$

where A is an integration constant. Formula (7) shows that these travelling waves are mainly characterized by the profile and the velocity. It is worthwhile noting that they are determined modulo phase shifts, since the one-parameter group of translations in space is a symmetry group of (1). This defines an orbit of solutions $\{\phi_c(x-x_0):x_0\in\mathbb{R}\}$ whose elements remain in the same level set $\{\varphi/V(\varphi)=V(\phi_c)\}$. The group parameter x_0 would play the role of the phase of the wave.

3. Description of the numerical method

Typically, these travelling waves are not explicitly known (important exceptions include the KdV and Benjamin–Ono equations) and then numerical approximation is necessary for their computation. As mentioned before, considering the simulation of the waves in a more geometric sense is suggested. One could approximate the orbit of the wave and focus on the elements that determine this orbit. This would imply an efficient way to compute the initial profile and the inclusion, in the numerical integration, of some properties in order to get a suitable simulation of the parameters.

To this end, and for further purposes, we will make some more assumptions on (1) [8]:

(H4) $f(\phi_c(x)) \geq 0$, $x \in \mathbb{R}$.
(H5) The linearized operator of (7) at ϕ_c,

$$L_c = c + M - f'(\phi_c),$$

has a unique, negative, simple eigenvalue, the zero eigenvalue is simple and the rest of its spectrum is bounded away from zero.

3.1. Spatial discretization

We begin the description of our proposal with the discretization in space. The nonlocal term in (1) makes the spectral-type methods a good choice for the spatial discretization. Here we approximate the solutions of the periodic problem of (1) by a Fourier pseudospectral discretization. First we give the description for the 2π-periodic problem and then we adapt the formulation for any interval of periodicity $(-L, L)$ [11–13]. For an even number N of nodes $x_j = -\pi + jh$, $h = 2\pi/N$, $j \in \mathbb{Z}$, we consider the space S_h of periodic functions $Z = \{Z_j\}_{j\in\mathbb{Z}}$ defined on the grid, with $Z_{j+N} = Z_j$. For each $Z \in S_h$, the discrete Fourier coefficients

$$\widehat{Z}_p = \frac{1}{N}\sum_{0\leq j\leq N}{}'' Z_j e^{-ipjh}, \quad -\frac{N}{2} \leq p \leq \frac{N}{2}, \tag{8}$$

provide the information of Z in the Fourier space. In (8), the double prime in the sum denotes that the first and last terms are divided by two. The reconstruction of Z from the Fourier coefficients is carried out by evaluating, at the grid points, the trigonometric interpolation polynomial

$$Z_h(x) = \sum_{-N/2\leq p\leq N/2}{}'' \widehat{Z}_p e^{ipx}, \tag{9}$$

in such a way that $Z_j = Z_h(x_j)$.

On the other hand, the pseudospectral differentiation operator on Z is obtained by differentiating (9) with respect to x and evaluating at the x_j:

$$(DZ)_j = \sum_{-N/2\leq p\leq N/2}{}'' \widehat{Z}_p(ip)e^{ipjh}, \quad j \in \mathbb{Z}.$$

In terms of the discrete Fourier coefficients, we have

$$\widehat{(DZ)}_p = (ip)\widehat{Z}_p, \quad -N/2 \leq p \leq N/2,$$

which means that, in the Fourier space, the operator D diagonalizes and differentiation is represented by the product with the diagonal matrix with elements ip, $-N/2 \leq p \leq N/2$.

With the Fourier pseudospectral method based on the x_j, the semidiscrete approximation to the 2π-periodic problem of (1) is a map $U : [0, \infty) \to S_h$ satisfying

$$\begin{aligned} &\frac{dU_j}{dt}(t) + (D(f(U)))_j(t) + M(DU)_j(t) = 0, \\ &U_j(0) = u(x_j, 0), \quad 0 \leq j \leq N-1, \end{aligned} \tag{10}$$

where:

- $U(t) = (U_0(t), \ldots, U_{N-1}(t))$ and $U_j(t)$ is an approximation to $u(x_j, t)$, $j = 0, \ldots, N-1$.
- The expression $f(U(t))$ denotes $(f(U_0(t)), \ldots, f(U_{N-1}(t)))$.

Observe that if $\widehat{U}_p(t)$ is the p-th discrete Fourier component of $U(t)$, the system (10) can be described in a more suitable form:

$$\begin{aligned} &\frac{d}{dt}\widehat{U}_p(t) = (ip)(\alpha(p)\widehat{U}_p(t) + \widehat{f(U)}_p(t)) \\ &\widehat{U}_p(0) = \widehat{u}_p(0), \end{aligned} \tag{11}$$

where $\widehat{u}_p(0)$ denotes the p-th discrete Fourier component of the initial condition $(u(x_0, 0), \ldots, u(x_{N-1}, 0))$. System (10) is then implemented in the form (11). It is also stiff, which will influence the choice of the time integrator [14]. On the other hand, the computation of the nonlinear term can be done by minimizing the generation of aliasing errors [11]. Also well known are the properties of convergence of the pseudospectral method, depending on the smoothness of the solution [13]. Finally, the connection with the $2L$-periodic problem of (11) requires one to transform the spatial variable into the form $X = \pi(x + L)/L$ and to write (1) with the corresponding scaling. In particular, the pseudospectral differentiation operator must be multiplied by the factor π/L.

3.2. Generation of the initial profile

The combination of the pseudospectral spatial discretization with the adaptation of Petviashvili's method in order to compute travelling-wave solutions establishes a technique for generating the initial profile.

Petviashvili's method [15] was originally implemented for computing solitary waves of the initial value problem for the KPI equation, and its application to stationary and solitary-wave solutions of other problems has also been proposed (see e.g. [16] and references therein). The method can be adapted to the periodic case as follows. Denoting by $\widehat{\phi}(k)$ the k-th Fourier coefficient of ϕ, Eq. (7) generates a system for the Fourier coefficients

$$(c + \alpha(k))\widehat{\phi}(k) - \widehat{f(\phi)}(k) = A\mathbb{I}(k), \quad k \in \mathbb{Z},$$

where $\mathbb{I}(k)$ denotes the k-th Fourier coefficient of the function $u = 1$. Then the Fourier coefficients of ϕ satisfy

$$\widehat{\phi}(k) = \frac{\widehat{f(\phi)}(k) + A\mathbb{I}(k)}{(c + \alpha(k))}, \quad k \in \mathbb{Z}. \tag{12}$$

Note that if we multiply (7) by ϕ and integrate in $(-L, L)$ then

$$K = K(\phi) = \frac{\int_{-L}^{L} \phi((c + M)\phi)\mathrm{d}x}{\int_{-L}^{L} \phi(A + f(\phi))\mathrm{d}x} = 1. \tag{13}$$

Eq. (12) suggests an iterative technique for the numerical approximation. In these cases, the classical fixed point iteration usually diverges and Newton-type methods are computationally more expensive. Petviashvili suggested an alternative by introducing a free parameter γ and a stabilizing factor K, so that the modified iterative scheme has the form

$$\widehat{\phi}(k)^{[\nu+1]} = K(\phi^{[\nu]})^{\gamma} \frac{\widehat{f(\phi^{[\nu]})}(k) + A\mathbb{I}(k)}{(c + \alpha(k))}, \quad k \in \mathbb{Z},\ \nu = 0, 1, \ldots, \tag{14}$$

where $\widehat{\phi}(k)^{[\nu]}$ stands for the ν-th iteration and γ is chosen to check the convergence of (14). In the case of the initial value problems and solitary wave solutions, local convergence is obtained under the assumptions (H1)–(H5) and for $\gamma \in (1, (p + 1)/(p - 1))$ [8]. Furthermore, the fastest rate of convergence occurs for $\gamma^* = p/(p - 1)$.

Having in mind the spatial discretization described in the previous subsection, the discrete version of the iterative procedure (14) can be written in terms of the discrete Fourier coefficients of the pseudospectral approximation to the profile ϕ:

$$\widehat{Z}_p^{[\nu+1]} = \widetilde{K}(Z^{[\nu]})^{\gamma} \frac{\widehat{f(Z^{[\nu]})}_p + A\mathbb{I}(p)}{(c + \alpha(p))}, \qquad -N/2 \le p \le N/2, \quad \nu = 0, 1, \ldots. \tag{15}$$

The stabilizing factor is approximated by a term $\widetilde{K}(Z)$ as follows. From the Parseval identity, (13) can be written in terms of the Fourier coefficients as

$$K(\phi) = \frac{\sum_{k=-\infty}^{\infty} (c + \alpha(k))|\widehat{\phi}(k)|^2}{\sum_{k=-\infty}^{\infty} (\widehat{f(\phi)}(k) + A\mathbb{I}(k))\overline{\widehat{\phi}(k)}}.$$

Then, for $Z \in S_h$, we define

$$\widetilde{K}(Z) = \frac{\sum_{p=-N/2}^{N/2} (c + \alpha(p))|\widehat{Z}_p|^2}{\sum_{p=-N/2}^{N/2} (\widehat{f(Z)}_p + A\mathbb{I}(p))\overline{\widehat{Z}_p}}. \tag{16}$$

3.3. Time integration

The third part of our proposal concerns the time integration. Here, a first criterion of selection of the time integrator may be the stiffness of (11). Then, the choice can be determined by the search for a good simulation of the velocity parameter

of the periodic travelling wave. Classical discretizations of (1) always include some properties of preservation of discrete versions of the invariants of the problem in the features of the numerical method. Recently, some results ([17] and references therein) show that a better simulation of the parameters of travelling-wave solutions is related to the preservation, through the numerical integration, of some invariants of the problem. Explicitly, the analysis of the time propagation of the error shows that this is affected by secular components, associated with the parameters of the wave. These secular terms behave better in those methods that preserve discretized versions of the invariants of the problem, providing a more suitable, in a qualitative sense, simulation of the travelling wave. In [17] this was studied for one of the equations included in (1), the KdV equation, and we conjecture that similar conclusions for the cases covered by (1), under the hypotheses (H1)–(H5), also hold.

Accordingly to this, it seems that preservation of (3)–(5) should be a desirable property for a time integrator in this context. By considering the pseudospectral spatial discretization, we introduce the discrete versions of the invariants

$$\widetilde{I}(Z) = h\sum_{j=0}^{N-1} Z_j, \tag{17}$$

$$\widetilde{V}(Z) = \frac{h}{2}\sum_{j=0}^{N-1} Z_j^2, \tag{18}$$

$$\widetilde{H}(Z) = h\sum_{j=0}^{N-1} \frac{1}{2}(Z_j(MZ)_j) - F(Z)_j, \tag{19}$$

for $Z \in S_h$.

With these requirements, and to treat the oscillatory problem, we first take the simply diagonally implicit Runge–Kutta (SDIRK) method of order 3 and the tableau

$$\begin{array}{c|cc} \frac{3+\sqrt{3}}{6} & \frac{3+\sqrt{3}}{6} & 0 \\ \frac{3-\sqrt{3}}{6} & \frac{-\sqrt{3}}{3} & \frac{3+\sqrt{3}}{6} \\ \hline & \frac{1}{2} & \frac{1}{2}. \end{array} \tag{20}$$

Other alternatives are also possible. The method has a good computational behaviour in the implicit systems for the intermediate stages to be solved at each step. However, (20) does not preserve discrete versions of the invariants (4) and (5). The preservation of (18), (19) will be forced by using the projection technique (see [18,19]). We can give a brief description of the method in the case of the preservation of $\widetilde{V}$ (the case of $\widetilde{H}$ would be similar). Assume that V_0 is the value of (4) at the initial profile, and let U_n be the numerical approximation to the solution $u(t)$ of (1) at some time discrete value t_n. Then the following approximation U_{n+1} is carried out in two steps:

(i) Compute $\tilde{U}_{n+1}$ by using (20).
(ii) Project the value $\tilde{U}_{n+1}$ onto the manifold

$$\mathcal{M}_0 = \{Z \in S_h / \tilde{V}(Z) = V_0\}.$$

In this second step a critical point condition of the Lagrange function for the corresponding constrained minimization problem has to be solved iteratively; see [18,19] for details.

Note that condition (6) establishes a relation between the variational derivatives of the invariants (3)–(5) for the initial profile. When simulating periodic travelling-wave solutions, this has two consequences. The first one is that we cannot implement a projection in order to preserve the three quantities at the same time. The second one is that a better performance is obtained when the method preserves two of the quantities, but there is no priority in the selection of the invariants to be conserved (see e.g. [17] for the details). When simulating perturbations of these periodic travelling waves or other periodic solutions of (1), then (6) does not hold and the situation may be different.

4. Numerical experiments

In order to illustrate the numerical performance of the method described previously, in this section we will consider the periodic Benjamin–Ono equation as the model problem. This equation is the case of (1) corresponding to $f(u) = u^2/2$ and $M = -\mathcal{H}\partial_x$ where $\mathcal{H}$ is the periodic Hilbert transform

$$\mathcal{H}u(x) = PV\frac{1}{2L}\int_{-L}^{L} \cot\left(\frac{\pi}{2L}y\right) u(x-y)\mathrm{d}y.$$

The operator M has the symbol $\alpha(k) = |k|$. Periodic travelling-wave solutions of this problem are explicitly known and have the form (see e.g. [20])

Table 1

Errors of the iterative method (15). The starting iterations are $Z_j = \phi_L(x_j) + \epsilon e^{-x_j^2}$ with $\epsilon = 1E{-}03$ (ERROR1) and $\epsilon = 1E{-}01$ (ERROR2).

Iteration	ERROR1	ERROR2
5	9.8374E−07	9.5179E−03
6	7.3680E−07	7.1284E−03
7	5.5415E−07	5.3611E−03
8	4.1783E−07	4.0423E−03

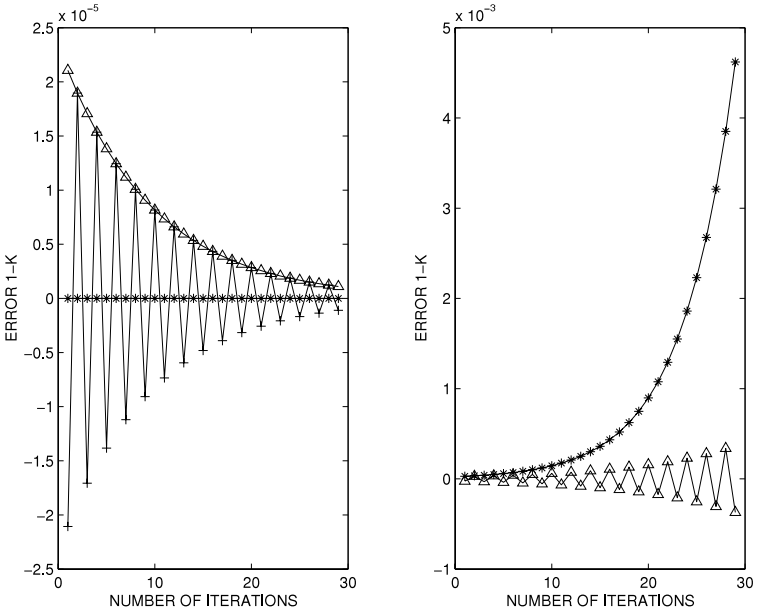

Fig. 1. Error in the computation of (16) against number of iterations. Left: $\gamma = 2(*)$, $\gamma = 2.9\,(+)$, $\gamma = 1.1\,(\triangle)$. Right: $\gamma = 0.8(*)$, $\gamma = 3.1\,(\triangle)$.

$$u_L(x,t) = \frac{2c\delta^2}{1-\sqrt{1-\delta^2}\cos(c\delta(x-ct-x_0))}, \quad x \in (-L,L),\ t>0, \tag{21}$$

with $c > 0$, $\delta = \pi/(cL)$, $x_0 \in \mathbb{R}$. The corresponding profile $\phi_L(x) = u(x,0)$ satisfies (7) with $A = 0$. In the following sections, we will take $L = 16$, $c = 1$ and $x_0 = 0$. All this information about the solutions will serve to illustrate the behaviour of the scheme. The numerical experiments, according to the main goals of the method (explained above), are focused on the computation of the profile and the simulation of the parameters.

4.1. Computation of the profile

As far as the first question is concerned, Table 1 shows, for two different starting iterations, the error in Euclidean norm between the exact profile at the points x_j and the corresponding numerical approximation given by the iteration (15), controlled by a maximum number of iterations and a tolerance for the relative error between two consecutive iterations. The starting profiles are small perturbations of the exact one, in the form $Z_j = \phi_L(x_j) + \epsilon e^{-x_j^2}$, with $\epsilon = 1E{-}03$ for the second column (ERROR1) and $\epsilon = 1E{-}01$ for the third column (ERROR2). The experiments are performed with $\gamma = 2$, which is optimal for the iteration in the case of the corresponding initial value problem [8]. The results show the convergence of the iteration, although the third column reveals its local character, since for a starting value which is not so close to the exact profile, the convergence is slower. In some cases, this local character of the convergence and, in particular, the choice of the starting iterations, may be improved by using some technique of numerical continuation (see e.g. [21] and references therein). This is particularly suitable when the equation is governed by some parameters. A typical example of this situation in our context is the Benjamin equation [22].

It is worth checking the influence of the parameter γ on the convergence of the iteration for this periodic case. Fig. 1 (left) shows the behaviour of the error in the stabilizing factor (16) as a function of the number of iterations, for the values $\gamma = 1.1, 2, 2.9$. In Fig. 1 (right), we take $\gamma = 0.8, 3.1$. Both are obtained with the initial data used in the second column of Table 1. We have already mentioned that, in order to ensure the local convergence, for the corresponding iteration in the initial value problem, γ must be in the range $(1, (p+1)/(p-1))$, which, in our case, is the interval $(1, 3)$, since we have $p = 2$ [8]. The numerical computations performed here suggest that this also happens in the case of the periodic problem. Fig. 1, left, shows this convergence, although it is of different type, depending on the value of γ taken in the interval $(1, 3)$,

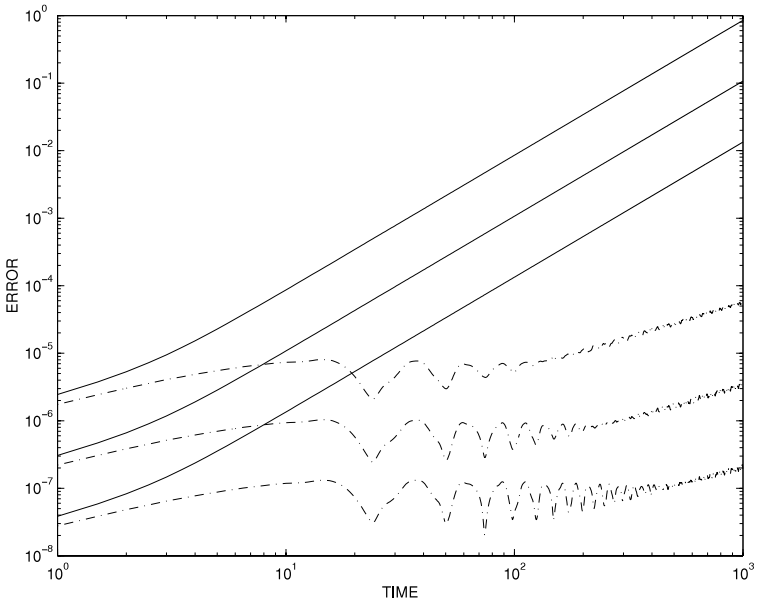

Fig. 2. Error vs. time on a log–log scale. Solid lines: SD. Dashed lines: CSD. The time steps are $\Delta t = 1/80,\ 1/160,\ 1/320$.

and the best one seems to correspond to $\gamma = 2$, as in the case of the initial value problem. For values of γ out of this interval, the stabilizing factor does not converge and the iteration procedure is divergent. This is observed in Fig. 1 (right).

4.2. Numerical simulation of the parameters

A second point that we have studied is the numerical evolution of the velocity. We have simulated the periodic problem with two time integrators: the method (20), denoted by SD, as an example of a nonconservative method, and the scheme (20) combined with a projection to preserve the quantities (17) and (18), denoted by CSD. A Hamiltonian preserving method gives similar results and will not be shown here (see the remarks in the previous section). In our computations, we have observed that, although SD preserves (17), the modification to force only the conservation of (18) gives a non-I-preserving method (see also [17]), and then a projection involving both quantities is necessary.

The knowledge of the exact solutions (21) allows us to measure the differences between these two time integrators, for a moderately long time of simulation, by comparing the exact values with the corresponding approximations. Fig. 2 shows, on a logarithmic scale, the evolution of the error given by the two methods, for different values of the time step, up to a final time $t = 10^3$. Solid lines correspond to SD and broken lines to CSD. The slope of the lines shows that, for the SD scheme, errors behave as t^2, the growth only being linear in the case of CSD. This reveals a better performance of the latter for long time simulations.

The most harmful components of the error, considered as a function of time, seem to be related to the parameters. This is suggested in Fig. 3. This shows, also on a log–log scale, the evolution of the error between the velocity of the numerical approximation and the exact one for the same experiments as in Fig. 2. The computation of the numerical velocity has been done in a standard way ([17] and references therein). The left figure corresponds to SD and the right figure to CSD. Note that, while the simulation of SD provides a computation of the velocity that grows linearly with time with respect to the exact one, the CSD method gives an approximation to the exact c that is constant in time, with no secular perturbations. Note also that, due to the relation given in (21), the simulation of c will affect, among other parameters, the amplitude of the numerical wave.

The experiments performed in this section for the Benjamin–Ono equation, and the theory explained in [17] for the KdV equation, suggest the following behaviour of the parameters of the numerical approximations to periodic travelling-wave solutions of the general problem (1) under the conditions (H1)–(H5). The explanation would say that the numerical solution contains a travelling-wave profile $U(x, t_n, c_n, x_{0,n})$ with the main source of the time propagation of the error. The parameters c_n and $x_{0,n}$ are perturbations of c and x_0 respectively, that evolve with time. This time evolution depends on the integrator used. In the general case, the dominant terms of $x_{0,n}$ contain time quadratic perturbations of the original phase, while the leading behaviour of c_n, when comparing with c, grows linearly with time. This would explain the performance of the SD method shown in Figs. 2 and 3. This behaviour may be improved when the method preserves discrete versions of the quantity (3) and one of the quantities (4) or (5). In this case, the leading term of the perturbation of c is constant in time and the numerical solution is only affected by a change of phase which grows linearly in time. This provides, in a qualitative sense, a better simulation for long times and a better approximation to the orbit of the travelling wave.

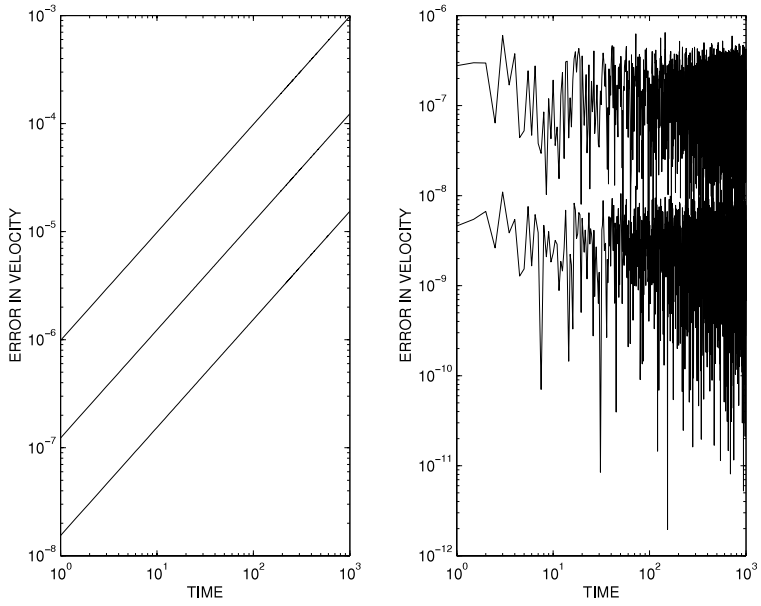

Fig. 3. Error in speed vs. time on a log–log scale. Left: SD with time steps $\Delta t = 1/80, 1/160, 1/320$. Right: CSD with $\Delta t = 1/80, 1/320$.

We have also carried out the same experiments as above but with an initial profile obtained after a convergent process using (15) for $\gamma = 2$ and a perturbation of the exact profile as the starting iteration. The numerical results in the simulation were similar to those shown here, which were obtained with the exact profile as the initial data. This suggests considering this combination of techniques, computing the profile and the parameters, as an efficient way to approximate periodic travelling-wave solutions of (1) with unknown analytical expression or to simulate perturbations of these waves.

Acknowledgements

This work was supported by: Ministerio de Ciencia e Innovación, under projects MTM2008-00700/MTM, MTM2008-01396-E, MTM2009-06507-E and MTM2010-19510/MTM; Junta de Castilla y León, under projects VA001A08 and VA060A09 and Consolider Ingenio Mathematica, under project SAIRT-C4-0189.

References

[1] L. Abdelouhab, J.L. Bona, M. Felland, J.C. Saut, Nonlocal models for nonlinear, dispersive waves, Physica D 40 (1989) 360–392.
[2] J.P. Albert, J.L. Bona, J.C. Saut, Model equations for stratified fluids, Proc. R. Soc. Lond. Ser. A 453 (1997) 1233–1260.
[3] J. Angulo Pava, Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions, in: Mathematical Surveys and Monographs, vol. 156, AMS, Providence, 2009.
[4] P.D. Lax, Periodic solutions of the KdV equation, Comm. Pure Appl. Math. 28 (1975) 141–188.
[5] J. Vigo-Aguiar, T.E. Simos, J.M. Ferrandiz, Controlling the error growth in long-term numerical integration of perturbed oscillations in one or several frequencies, Proc. R. Soc. Lond. Ser. A 460 (2004) 561–567.
[6] T.E. Simos, J. Vigo-Aguiar, A new modified Runge–Kutta–Nystrom method with phase-lag of order infinity for the numerical solution of the Schödinger equation and related problems, Internat. J. Modern Phys. C 11 (2000) 1195–1208.
[7] T.E. Simos, J. Vigo-Aguiar, A modified Runge–Kutta–Nystrom method with phase-lag of order infinity for the numerical solution of the Schödinger equation and related problems, Comput. Chem. 25 (2001) 275–281.
[8] D.E. Pelinovsky, Y.A. Stepanyants, Convergence of Petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations, SIAM J. Numer. Anal. 42 (2004) 1110–1127.
[9] T.B. Benjamin, J.L. Bona, D.K. Bose, Solitary-wave solutions of nonlinear problems, Philos. Trans. R. Soc. Lond. Ser. A 331 (1990) 195–244.
[10] H. Chen, Existence of periodic travelling-wave solutions of nonlinear, dispersive wave equations, Nonlinearity 17 (2004) 2041–2056.
[11] J.P. Boyd, Chebyshev and Fourier Spectral Methods, 2nd ed., Dover Publications, New York, 2000.
[12] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer-Verlag, New York, Heidelberg, Berlin, 1988.
[13] J. Pasciak, Spectral methods for a nonlinear initial value problem involving pseudo-differential operators, SIAM J. Numer. Anal. 19 (1982) 142–154.
[14] J. Vigo-Aguiar, S. Natesan, A parallel boundary value technique for singularly perturbed two-point boundary value problems, J. Supercomput. 27 (2004) 195–206.
[15] V.I. Petviashvili, Equation of an extraordinary soliton, Sov. J. Plasma Phys. 2 (1976) 257–258.
[16] T.I. Lakoba, J. Yang, A generalized Petviashvili method for scalar and vector Hamiltonian equations with arbitrary form of nonlinearity, J. Comput. Phys. 226 (2007) 1668–1692.
[17] A. Durán, Time behaviour of the error when simulating finite-band periodic waves. The case of the KdV equation, J. Comput. Phys. 227 (2008) 2130–2153.
[18] E. Hairer, S.P. Norsett, G. Wanner, Solving Ordinary Differential Equations I. Nonstiff Problems, 2nd ed., in: Springer Series in Computational Mathematics, vol. 8, Springer-Verlag, New York, Heidelberg, Berlin, 1993.
[19] E. Hairer, C. Lubich, G. Wanner, Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations, Springer-Verlag, New York, Heidelberg, Berlin, 2002.
[20] V. Thomée, A.S. Vasudeva Murthy, A numerical method for the Benjamin–Ono equation, BIT 38 (1998) 597–611.
[21] E.L. Allgower, K. Georg, Numerical Continuation Methods: An Introduction, Springer, New York, 1990.
[22] J.P. Albert, J.L. Bona, J.M. Restrepo, Solitary-wave solutions of the Benjamin equation, SIAM J. Appl. Math. 59 (1999) 2139–2161.